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The path integral quantization and the construction of the S-matrix operator in the Abelian and non-Abelian Chern–Simons theories

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Abstract. The covariant path integral quantization of the theory of the scalar and spinor fields interacting through the Abelian and non-Abelian Chern–Simons gauge fields in $2+1$ dimensions is carried out using the De Witt–Fadeev–Popov method. The mathematical ill-definiteness of the path integral of theories with pure Chern–Simons' fields is remedied by the introduction of the Maxwell or Maxwell-type (in the non-Abelian case) terms, which make the resulting theories super-renormalizable and guarantees their gauge-invariant regularization and renormalization. The generating functionals are constructed and shown to be the same as those of quantum electrodynamics (quantum chromodynamics) in $2+1$ dimensions with the substitution of the Chern–Simons propagator for the photon (gluon) propagator. By constructing the propagator in the general case, the existence of two limits; pure Chern–Simons and quantum electrodynamics (quantum chromodynamics) after renormalization is demonstrated.

The Batalin–Fradkin–Vilkovisky method is invoked to quantize the theory of spinor non-Abelian fields interacting via the pure Chern–Simons gauge field and the equivalence of the resulting generating functional to the one given by the De Witt–Fadeev–Popov method is demonstrated.

The S-matrix operator is constructed, and starting from this S-matrix operator novel topological unitarity identities are derived that demand the vanishing of the gauge-invariant sum of the imaginary parts of the Feynman diagrams with a given number of intermediate on-shell topological photon lines in each order of perturbation theory. These identities are illustrated by explicit examples.

1. Introduction

The past 15 years witnessed an increasing interest in the theories of matter coupled Chern–Simons (CS) gauge field theories in $2+1$ dimensions. From one point of view, the Euclidean version of such theories can be viewed as giving the high temperature bahaviour of $3+1$ dimensional models [1]. On the other hand, in the pioneering works [2, 3] it has been shown that the introduction of the (P and T odd) CS term into the Lagrangian of $2+1$ dimensional quantum electrodynamics (QED) and quantum chromodynamics (QCD), leads to a very peculiar property: the gauge field splits into two parts; a massive part (that acquires a mass in a gauge-invariant manner), and a massless part which does not contribute to the free classical Hamiltonian, but leads to an additional interaction among the particles. This interaction also appears in pure CS theories [4].

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In the work [3], it was argued that in the non-Abelian version of CS theories, the dimensionless combination of the charge and the stochastic parameter should be quantized. It was also shown that the mass term provides an infrared cut-off in special covariant gauges that renders the theory superrenormalizable.

Many works were devoted to the consideration of the one-loop radiative corrections to the charge and the stochastic parameter in both the Abelian [5] and the non-Abelian [6] theories, and a theorem [7] was set which states that under very general conditions, there are no further radiative corrections beyond the finite one-loop for these parameters.

An additional thrust into the interest in CS theories was provided by the interesting results in the non-relativistic domain; essentially the idea of Wilczeck that non-relativistic charged particles coupled to pure CS fields can be considered as a phenomenological approach for the description of the 'bound states' of two particles called anyons [8]. This idea found wide acceptance, and many attempts to apply it in many interesting condensed matter phenomena, such as the fractional quantum Hall effect, and high-temperature superconductivity were made (see the reviews [9] and the references therein). CS theories also found applications in the field-theoretic formulation of the Aharonov–Bohm effect [10, 11].

One of the issues that received considerable interest during the past period was the canonical quantization of the CS models [2, 3, 12]. However, some interesting points such as the canonical quantization in a Lorentz covariant gauge still need further investigation. Path integral quantization was also considered first—to our knowledge—in the works [11, 13] where the generating functional was also constructed.

Another issue that did not receive much attention is the following. The free transverse topological photons of the pure CS theory are absent, while the gauge-field propagator is present, and significantly contributes to the interaction among the particles. This issue was addressed in the work [14], and the so-called topological unitarity identities were derived. We elaborate on this issue in the present work.

This paper is a further development of the series of works [11, 13, 14]. The main aims are, to carry out the path integral quantization and construct the generating functional for a wide class of models involving both the Abelian and the non-Abelian CS fields (section 2), to construct the *S*-matrix operator, and to develop the Feynman rules and formulate a Wick-type theorem for the CS field (section 3), and to illustrate in detail the topological unitarity identities in general, and through a specific example (section 4). Section 5 is devoted to concluding remarks.

At this point, we would like to define some terms and abbreviations that we are going to use frequently later. By Chern–Simons quantum electrodynamics (CSQED), we mean QED with both the CS and the Maxwell terms present in the action. When only the CS term is present in the action, we refer to this as pure CSQED. In the non-Abelian case, we use CSQCD and pure CSQCD to refer to the theories with and without the Maxwell-type term, respectively.

2. Path integral quantization and the generating functional

The aim of this part is to develop the path integral quantization, and to construct the generating functional of the theory of scalar and spinor fields interacting through the Abelian and non-Abelian CS field in $2 + 1$ dimensions. This can be done through two different approaches: The De Witt–Fadeev–Popov (DFP) [15] approach, or the Batalin–Fradkin–Vilkovisky (BFV) approach [16]. The latter was developed to quantize gauge theories with both classes of constraints and with arbitrary constraint algebra. In our case both approaches lead to the same result. This is a consequence of the fact that the first-class constraints, both in the Abelian and non-Abelian cases, form a closed algebra, and that the structure functions in the algebra of the first-class constraints are just constants, as will be demonstrated later. Therefore, we shall carry out the path integral quantization through the simpler DFP approach, and will prove the equivalence of both approaches by invoking the latter in the quantization of the theory of spinors interacting through the non-Abelian CS gauge field. This proof is very helpful in understanding the connection between the usual canonical quantization and the BFV quantization schemes, and in the demonstration of the appearance of the BRST operators of the theory.

2.1. The De Witt–Fadeev–Popov method

2.1.1. The scalar CSQED. We begin with the theory of charged scalar particles interacting through a gauge field whose action is given by both the Abelian CS and the Maxwell terms. Following the DFP method, we get for the generating functional in the covariant α-gauge the expression [11, 13]:

$$Z[J_\mu, j, j^*] = Z_0^{-1} \int DA_\mu(x) \mathrm{d}\varphi^*(x) D\varphi(x) \exp\left\{ \mathrm{i}S_{CS} + \mathrm{i}S_g + \mathrm{i}S_m + \mathrm{i}S_M \right. \\ \left. + \mathrm{i} \int \mathrm{d}^3x \, (J_\mu(x) A^\mu(x) + j^*(x)\varphi(x) + j(x)\varphi^*(x)) \right\} \tag{1}$$

where

$$Z_0 = Z(0, 0, 0,) \tag{2}$$

$$S_{CS} = \frac{\mu}{2} \int \mathrm{d}^3x \, \varepsilon_{\mu\nu\lambda} A^\mu(x) \partial^\nu A^\lambda(x) \tag{3}$$

$$S_g = \frac{-1}{2\alpha} \int (\partial_\mu A^\mu)^2 \, \mathrm{d}^3x \tag{4}$$

$$S_m = \int \mathrm{d}^3x \, (\varphi^*(x)(D_\mu D^\mu - m^2)\varphi(x) - \lambda(\varphi^*(x)\varphi(x))^2) \tag{5}$$

$$S_M = \frac{-1}{4\gamma} \int \mathrm{d}^3x \, F_{\mu\nu}(x) F^{\mu\nu}(x). \tag{6}$$

Here, $J_\mu(x)$, $j(x)$ and $j^*(x)$ are external sources, e and m are, respectively, the charge and the mass of the scalar field, and $D_\mu = (\partial_\mu - \mathrm{i}eA_\mu)$. The metric is taken as $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$.

The introduction of the Maxwell term (equation (6)) into the action of the theory guarantees the convergence of the path integral. This is because the latter is mathematically ill-defined when only the CS term is present in the action, since this term is not positive definite in Euclidean space. The Maxwell term is the only gauge-invariant bilinear term in A_μ that guarantees gauge-invariant regularization and renormalization of the theory. This term not only leads to the convergence of the functional integral over A_μ, but also plays the role of a regularization factor since the resulting theory becomes super-renormalizable [2, 3]. The pure CS theory can be recovered by taking the limit $\gamma \to \infty$ as will be shown below.

The Green functions of the theory are defined as usual by varying the above generating functional, equation (1), with respect to the sources. For example, the free propagator of the CS field is defined as

$$D_{\mu\nu}(x - x') = (-\mathrm{i})^2 \frac{\delta^2}{\delta J_\mu(x) J_\nu(x')} Z[J_\mu, j, j^*] \,|_{J_\mu = j = j^* = e = 0} . \tag{7}$$

Before proceeding further, it is necessary here to make some remarks on the dimensions of the parameters and the fields of the theory. We have some arbitrariness in the choice of the dimensions of the statistical parameter μ, the charge e and the factor γ in equations (3), (5) and (6). However, if we require the $2+1$ dimensional matter-coupled CS theory to have some relation with the real world, so that it arises after compactification on the $\sim \frac{1}{\gamma}$ layer of QED in $3+1$ dimensions [17] with the parity violating term $\frac{\mu}{4}\int F_{\mu\nu}\tilde{F}^{\mu\nu}\,\mathrm{d}^4x$ where $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ then the charge e and the parameter μ are to be chosen dimensionless, whereas $[A_\mu] = x^{-1}$, $[\varphi] = x^{\frac{-1}{2}}$ and $[\gamma] = x^{-1}$. In the following, we will adopt this convention of the dimensions†.

The generating functional, equation (1), can be formally rewritten in the alternative form

$$Z[J_\mu, j^*, j] = Z_0^{-1}\int D\varphi^*(x)D\varphi(x)\exp\left(\mathrm{i}e^2\int \mathrm{d}^3x\,\frac{\delta^2}{\delta J_\mu(x)\delta J^\mu(x)}\right)$$
$$\times\int DA_\mu(x)\exp\Bigg\{\mathrm{i}(S_{CS}+S_M+S_g+\tilde{S}_m)$$
$$+\mathrm{i}\int\mathrm{d}^3x\,(J_\mu(x)A^\mu(x)+j^*(x)\varphi(x)+\varphi^*(x)j(x))\Bigg\} \tag{8}$$

where $\tilde{S}_m$ does not contain the term $e^2A_\mu A^\mu$ in equation (5), i.e.

$$\tilde{S}_m = -\int\mathrm{d}^3x\,(\mathrm{i}eA_\mu(x)(\varphi^*(x)\partial_\mu\varphi(x)-\varphi(x)\partial_\mu\varphi^*(x))+\lambda(\varphi^*(x)\varphi(x))^2). \tag{9}$$

After integrating over A_μ in equation (8) we obtain:

$$Z[J_\mu, j, j^*] = Z_0^{-1}\int D\varphi^*(x)D\varphi(x)\exp\left\{\mathrm{i}e^2\int\mathrm{d}^3x\,\varphi^*(x)\varphi(x)\frac{\delta^2}{\delta J_\nu(x)\delta J^\nu(x)}\right\}$$
$$\times\exp\Bigg\{\frac{\mathrm{i}}{2}\int\mathrm{d}^3x\,\mathrm{d}^3y\,I_\mu(x)D^{\mu\nu}(x-y)I_\nu(y)-\lambda\int\mathrm{d}^3x\,(\varphi^*(x)\varphi(x))^2$$
$$+\mathrm{i}\int\mathrm{d}^3x\,(j^*(x)\varphi(x)+j(x)\varphi^*(x))\Bigg\} \tag{10}$$

where

$$I_\mu(x) = J_\mu(x)+\mathrm{i}e\int\mathrm{d}^3x\,(\varphi^*(x)\partial_\mu\varphi(x)-\varphi(x)\partial_\mu\varphi^*(x)) \tag{11}$$

and $D_{\mu\nu}(x-y)$ is the CS gauge field's Green function (or propagator) defined by the equation:

$$\int\mathrm{d}^3x'\,\frac{\delta^2(S_{CS}+S_g+S_M)}{\delta A^\mu(x)\delta A^\lambda(x')}D^{\lambda\nu}(x'-y) = g_\mu^\nu\delta^3(x-y) \tag{12}$$

or,

$$\left[\frac{1}{\gamma}(\Box_x g_{\mu\lambda}-\partial_\mu\partial_\lambda)+\frac{1}{\alpha}\partial_\mu\partial_\lambda+\mu\varepsilon_{\mu\lambda\rho}\partial_x^\rho\right]D^{\lambda\nu}(x-y) = \delta^3(x-y)g_\mu^\nu. \tag{13}$$

The solution of equation (13) is [2, 3]:

$$D_{\lambda\nu}(x) = \frac{1}{(2\pi)^3}\int\mathrm{d}^3p\,\mathrm{e}^{\mathrm{i}px}\Bigg[-\gamma\frac{(g_{\lambda\nu}-\frac{p_\nu p_\lambda}{p^2})}{(p^2-\gamma^2\mu^2+\mathrm{i}\epsilon)}$$
$$+\frac{\mathrm{i}\varepsilon_{\lambda\nu\rho}p^\rho}{\mu(p^2-\gamma^2\mu^2+\mathrm{i}\epsilon)}-\frac{\mathrm{i}\varepsilon_{\lambda\nu\rho}p^\rho}{\mu(p^2+\mathrm{i}\epsilon)}-\frac{\alpha p_\lambda p_\nu}{(p^2+\mathrm{i}\epsilon)^2}\Bigg]. \tag{14}$$

† If one makes the change of variables $A_\mu \to A'_\mu = \frac{A_\mu}{\sqrt{\gamma}}$, $e \to e' = e\sqrt{\gamma}$, $\mu \to \mu' = \frac{\mu}{\gamma}$ then one gets the conventions used in the works [2, 3].

We note that the above Green function consists of two parts. The first two terms describe the propagation of a real massive photon with mass equal to $\gamma\mu$; the third term describes the propagation of a topological massless photon, and the last term is pure gauge term. The appearence of massive photons in a gauge-invariant manner is a well known peculiar property of CS theory, and is independent of coupling to matter fields [2, 3]. To show that the topological term in equation (14) does not contribute to the tensor $F_{\mu\nu}$ of the gauge field, we construct the general solution of the classical equations of motion of the field A_μ (equation (13)). This is given as:

$$\begin{aligned} A_\mu(x) &= 4\pi \int \operatorname{Im} D_{\mu\nu}(p) e^\nu_\delta a^\delta(p) \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}^3 p \\ &= \frac{1}{2\pi} \int \mathrm{d}^3 p\, \mathrm{e}^{\mathrm{i}px} \left[-\gamma \left(\left(e^\delta_\mu(p) - \frac{p_\mu p_\nu}{p^2} e^\nu_\delta(p) \right) + \frac{\mathrm{i}}{\gamma\mu} \varepsilon_{\mu\nu\rho} p^\rho e^\nu_\delta(p) \right) \right. \\ &\quad \times \delta(p^2 - \mu^2\gamma^2) - \frac{\mathrm{i}}{\mu} \varepsilon_{\mu\nu\rho} e^\nu_\delta(p) p^\rho \delta(p^2) - \left(\frac{p_\mu p_\nu}{p^2 - \mu^2\gamma^2} \right) e^\nu_\delta(p) \delta(p^2) \\ &\quad \left. + \frac{\alpha}{2} p_\mu \left(\frac{\partial}{\partial p^\nu} \delta(p^2) \right) e^\nu_\delta(p) \right] a^\delta(p). \end{aligned} \tag{15}$$

Here, $\operatorname{Im} D_{\mu\nu}(p)$ is the imaginary part of the propagator $D_{\mu\nu}$ in equation (14) in the momentum space representation; $e^\nu_\delta(p), \delta = 0, 1, 2,$ are three mutually orthogonal polarization vectors which satisfy $p_\mu e^\mu_\delta(p) = 0$. This choice corresponds to the gauge $\partial_\mu A^\mu = 0$. In the general case, the free solution $A_\mu(x)$ in equation (15) represents the sum of two independent parts. The terms proportional to $\delta(p^2 - \mu^2\gamma^2)$ correspond to a real massive photon which contributes to the free Hamiltonian; the fourth and fifth terms are the topological parts of the gauge field which do not contribute to the classical free Hamiltonian, but give non-trivial contribution to the propagator (see equation (14)), and the last term is merely a gauge term that can be removed by a gauge transformation. It is easy to see that the topological part of A_μ does not contribute to $F_{\mu\nu}$:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \frac{1}{2\pi\mu} \int \mathrm{d}^3 p\, \mathrm{e}^{\mathrm{i}px} \delta(p^2) a^\delta(p) (p_\mu \varepsilon_{\nu\lambda\rho} - p_\nu \varepsilon_{\mu\lambda\rho}) e^\lambda_\delta p^\rho \end{aligned} \tag{16}$$

multiplying both sides by $\varepsilon_{\sigma\mu\nu}$ we get

$$\varepsilon^{\sigma\mu\nu} F_{\mu\nu} = \frac{1}{\mu\pi} \int \mathrm{d}^3 p\, \mathrm{e}^{\mathrm{i}px} \delta(p^2) a^\delta(p) (e^\sigma_\delta(p) p^2 - p_\mu e^\mu_\delta p^\sigma) = 0 \tag{17}$$

since

$$p_\mu e^\mu_\alpha = 0. \tag{18}$$

As for the the massive part of the solution (15), excluding the second term in this equation in view of (18) above, then we have for the massive part

$$A_\mu(x) = \frac{-1}{2\pi} \int \mathrm{d}^3 p\, \mathrm{e}^{\mathrm{i}px} \gamma \left(e^\delta_\mu(p) - \frac{\mathrm{i}}{\mu\gamma} \varepsilon_{\mu\nu\rho} p^\rho e^\nu_\delta \right) a_\delta(p) \delta(p^2 - \mu^2\gamma^2) \tag{19}$$

and this gives a non-vanishing contribution to $F_{\mu\nu}$. We shall return later to the question of quantization of this A_μ in connection with the construction of the S-matrix of the theory (see section 3).

Returning to the general expression for the Green function of the gauge field, we stress that formally it is possible to consider two limiting procedures in equation (14). First, if

$\gamma \to \infty$ we obtain:

$$\lim_{\gamma\to\infty} D_{\lambda\nu} = D^{CS}_{\lambda\nu} = \frac{-1}{(2\pi)^3}\int \mathrm{d}^3p\, \mathrm{e}^{\mathrm{i}px}\left(\frac{\mathrm{i}\varepsilon_{\nu\lambda\rho}p^\rho}{\mu(p^2+\mathrm{i}\epsilon)} + \frac{\alpha p_\lambda p_\nu}{(p^2+\mathrm{i}\epsilon)^2}\right) \tag{20}$$

which is just the propagator of the pure CS theory. In the limit $\mu \to 0$, we get the usual Feynman propagator in $2+1$ dimensional QED for massless photons:

$$\lim_{\mu\to 0} D_{\lambda\nu}(x) = D^M_{\lambda\nu}(x) = \frac{-\gamma}{(2\pi)^3}\int \mathrm{d}^3p\, \mathrm{e}^{\mathrm{i}px}\frac{\left(g_{\lambda\nu} - (\frac{p_\lambda p_\nu}{p^2+\mathrm{i}\epsilon})(1-\frac{\alpha}{\gamma})\right)}{(p^2+\mathrm{i}\epsilon)}. \tag{21}$$

In both cases, we have from equation (15) A_μ as

$$\begin{aligned}\lim_{\gamma\to\infty} A_\mu(x) &= \frac{-1}{2\pi}\int \mathrm{d}^3p\, \mathrm{e}^{\mathrm{i}px}\left[\left(\frac{\mathrm{i}}{\mu}\varepsilon_{\mu\nu\rho}p^\rho - \frac{\alpha}{2}p_\mu\frac{\partial}{\partial p^\nu}\right)\delta(p^2)\right]e^\nu_\delta(p)a^\delta(p)\\ &= A^{CS}_\mu\end{aligned} \tag{22}$$

and

$$\lim_{\mu\to 0} A_\mu(x) = \frac{-\gamma}{2\pi}\int \mathrm{d}^3p\, \mathrm{e}^{\mathrm{i}px}\left[\left(g_{\mu\nu} + \frac{(1-\frac{\alpha}{\gamma})}{2}p_\mu\frac{\partial}{\partial p^\nu}\right)\delta(p^2)\right]e^\nu_\delta(p)a^\delta(p). \tag{23}$$

Strictly speaking, the above limits are to be taken after renormalization. The parameter μ is known to receive finite renormalization at one-loop order [2], and the limit $\mu \to \infty$ at this order exists. Since it is well known that there are no further corrections from higher orders [5–7] for this parameter, then this limit exists to all orders in perturbation theory. The limit $\gamma \to \infty$, has recently been shown to exist up to two-loop order by Tan *et al* [23], who calculated the effective potential of the CS scalar electrodynamics with a symmetry-breaking term up to two-loop order, and showed that the limit $\gamma \to \infty$ exists, at which one recovers the pure CS theory.

2.1.2. The spinor CSQED. Let us now consider spinor CSQED. The DFP method gives the following expression for the generating functional in this case [14]:

$$\begin{aligned}Z[J_\mu,\eta,\bar\eta] = Z_0^{-1}\int DA_\mu(x)D\bar\psi(x)D\psi(x)\exp\Big\{&\mathrm{i}S_{CS} + \mathrm{i}S_M + \mathrm{i}S_g + \mathrm{i}S_\psi\\ &+\mathrm{i}\int \mathrm{d}^3x\,(J_\mu(x)A^\mu(x) + \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x))\Big\}\end{aligned} \tag{24}$$

where $Z_0 = Z(0,0,0)$; S_{CS}, S_g and S_M are defined by equations (3), (4) and (6) respectively, and

$$S_\psi = \int \mathrm{d}^3x\,\bar\psi(x)(\mathrm{i}\not{D} - m)\psi(x) \tag{25}$$

where

$$\not{D} = D_\mu\gamma^\mu \qquad D_\mu = (\partial_\mu - \mathrm{i}eA_\mu) \tag{26}$$

and the Dirac matrices are defined as

$$\gamma_0 = \sigma_3 \qquad \gamma_i = \mathrm{i}\sigma_i \qquad i = 1,2 \tag{27}$$

where σ's are the Pauli spin matrices. The γ-matrices satisfy

$$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu} \qquad \gamma_\mu\gamma_\nu = g_{\mu\nu} - \mathrm{i}\varepsilon_{\mu\nu\lambda}\gamma^\lambda \tag{28}$$

$\psi(x)$ and $\bar{\psi}(x) = \psi(x)^\dagger\gamma_0$ are the two-component Grassmann spinors, η and $\bar{\eta}$ are Grassmann sources. Integrating over $A_\mu(x)$ in equation (24) we obtain:

$$Z[J_\mu, \bar{\eta}, \eta] = Z_0^{-1} \int D\bar{\psi}(x) D\psi(x) \exp\left\{ \frac{\mathrm{i}}{2} \int \mathrm{d}^3x\, \mathrm{d}^3y\, \tilde{I}_\mu(x) D^{\mu\nu}(x-y) \tilde{I}_\nu(y) \right. \\ \left. +\mathrm{i} \int \mathrm{d}^3x\, (\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)) \right\} \tag{29}$$

where

$$\tilde{I}_\mu(x) = J_\mu(x) + e\bar{\psi}(x)\gamma_\mu\psi(x) \tag{30}$$

and $D_{\mu\nu}(x-y)$ is the bare CS field propagator which is the same as in the scalar case, equation (14). Here also, as in the scalar case, one can consider the limits (after renormalization) $\gamma \to \infty$ and $\mu \to 0$ to get the propagators of pure CS field and $2+1$ dimensional QED respectively. Moreover, the generating functionals equations (10) and (29) are, respectively, identical to those of scalar and spinor QED in $2+1$ dimensions with the substitution of the photon propagator for the CS propagator.

2.1.3. The non-abelian CS gluodynamics. The path integral quantization of theories with the non-Abelian CS gauge field is a bit more complicated than the Abelian one, so we consider it in some more detail. We start with the theory of the gauge field without coupling to matter, i.e. CS gluodynamics, defined by the Lagrangian

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_{CS} \tag{31}$$

$\mathcal{L}_M$ is the usual Yang–Mills Lagrangian in $2+1$ dimensions,

$$\mathcal{L}_M = \frac{-1}{2\gamma} \operatorname{tr}(F_{\mu\nu}(x)F^{\mu\nu}(x)) \\ F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + g[A_\mu(x), A_\nu(x)]. \tag{32}$$

$\mathcal{L}_{CS}$ is the non-Abelian CS term

$$\mathcal{L}_{CS} = -\mu\varepsilon^{\mu\nu\lambda} \operatorname{tr}\left(A_\mu(x)\partial_\nu A_\lambda(x) + \frac{2\mathrm{i}}{3} g A_\mu(x) A_\nu(x) A_\lambda(x) \right). \tag{33}$$

The gauge group is $SU(N)$. In matrix notation

$$A_\mu = A_\mu^a t^a \qquad F_{\mu\nu} = F_{\mu\nu}^a t^a. \tag{34}$$

The t^a's are anti-Hermitian matrices in the fundamental representation of the group

$$[t^a, t^b] = \mathrm{i} f^{abc} t_c \qquad \operatorname{tr}(t^a t^b) = \tfrac{1}{2}\delta^{ab}. \tag{35}$$

f^{abc} are the structure constants of the $SU(N)$ group.

To see the difference of the non-Abelian case from the Abelian one, consider a general gauge transformation

$$A_\mu(x) \to U^{-1}\left(A_\mu(x) - \frac{\mathrm{i}}{g}\partial_\mu \right) U. \tag{36}$$

$\mathcal{L}_M$ is gauge-invariant, $\mathcal{L}_{CS}$ is not [2, 3];

$$\int \mathrm{d}^3x\, \mathcal{L}_{CS} \to \int \mathrm{d}^3x\, \mathcal{L}_{CS} - \frac{\mathrm{i}\mu}{g} \int \mathrm{d}^3x\, \varepsilon^{\mu\nu\lambda} \partial_\mu \operatorname{tr}((\partial_\nu U) U^{-1} A_\lambda) + \frac{8\pi^2\mu}{g^2} \mathrm{i}w \tag{37}$$

where

$$w = \frac{1}{24\pi^2} \int \mathrm{d}^3x\, \varepsilon^{\mu\nu\lambda} \operatorname{tr}[(U^{-1}\partial_\mu U)(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U)]. \tag{38}$$

If we suppose that at $\|x\| = \sqrt{x_0^2 + \boldsymbol{x}^2} \to \infty$, $A_\mu \to 0$ faster than $\frac{1}{\|x\|}$ then the second term in (37) vanishes. The last term, however, coincides in *Euclidean* space, with the so-called homotopy class or winding number, and is equal to $0, \pm 1, \pm 2, \ldots$. This result follows from the fact that if

$$U(x)_{\|x\| \to \infty} \to 1 \tag{39}$$

then three-dimensional space can be mapped onto S_3; for $SU(2)$ group $U(x)$ realizes the mapping $S_3 \to S_3$ and the winding number is equal to the degree of mapping S_3 to the $SU(2)$ group. On the classical level, the gauge-transformation (36) results in

$$S_{CS} \to S_{CS} + \text{constant}. \tag{40}$$

It is clear that this constant does not influence the equations of motion or any physical quantity.

Now, we use the Fadeev–Popov trick to quantize the theory. Formally, the vacuum functional of the theory is

$$Z_0 = N \int DA_\mu \exp \mathrm{i}\{S_M + S_{CS}\} \tag{41}$$

where N is a normalization factor that will be defined later. Introducing into the formal equation (41) the identity operator in a general Lorentz covariant gauge

$$I = \triangle(A) \int D\mu(G) \delta(\partial^\mu A_\mu^G - f(x)) \tag{42}$$

where $D\mu(G)$ is the measure of the $SU(N)$ group, and

$$A_\mu^G = U^{-1} \left(A_\mu - \frac{\mathrm{i}}{g} \partial_\mu \right) U \qquad U \in G. \tag{43}$$

Equation (42) defines the Fadeev–Popov determinant $\triangle(A)$.

We know that in perturbation theory we can forget about the Gribov ambiguity [18] and consider only contributions to the functional integral from elements near the identity of the group G;

$$U = 1 + \mathrm{i}\lambda(x) + \mathrm{O}(\lambda^2) \qquad \lambda = \lambda^a t^a \tag{44}$$

where $\lambda^a(x)$ is infinitesimally small for all x. This means that in the DFP method we must consider only small gauge transformations which, by default, belong to the zero homotopy class for which $w = 0$ since $\lambda(x)$ must go to zero when $\|x\| \to \infty$†. Substituting the identity operator (42) into the expression (41), we get after the conventional manipulations

$$Z_0 = N\Omega(G) \int DA_\mu(x) D\bar{\mathcal{C}}(x) D\mathcal{C}(x) \exp\{\mathrm{i}(S_M + S_{CS} + S_g)\}. \tag{45}$$

Here $\Omega(G)$ is the infinite group volume, and

$$S_g = \int \mathrm{d}^3x \ \mathrm{tr} \left(\frac{-1}{2\alpha} (\partial_\mu A^\mu(x))^2 + \partial_\mu \bar{\mathcal{C}}^a(x) (D^{\mu ab} \mathcal{C}^b(x)) \right) \tag{46}$$

where $\mathcal{C}(x)$ and $\bar{\mathcal{C}}(x)$ are the well known Fadeev–Popov ghosts that are scalar Grassmann fields, and

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + g f^{abc} A_\mu^c(x). \tag{47}$$

† In the general case when $U = \mathrm{e}^{\mathrm{i}\tau^a \lambda^a(x)}$ and $\|x\| \to \infty$, $\lambda(x) = \sqrt{(\lambda^a)^2} \to 2\pi n$ where n is the winding number.

Thus, the generating functional of the theory is now given by the expression:

$$Z[J_\mu,\eta,\bar{\eta}] = Z^{-1}(0,0,0)\int DA_\mu(x)D\bar{\mathcal{C}}(x)D\mathcal{C}(x)\exp\Big\{\mathrm{i}(S_M+S_{CS}+S_g) + \mathrm{i}\int \mathrm{d}^3x\,(J^a_\mu(x)A^\mu_a(x)+\bar{\eta}^a(x)\mathcal{C}^a(x)+\bar{\mathcal{C}}^a(x)\eta^a(x))\Big\} \tag{48}$$

here

$$Z(0,0,0) = Z(J_\mu,\bar{\eta},\eta)\mid_{J_\mu=\bar{\eta}=\eta=0}. \tag{49}$$

Finally, we note that the action in the generating functional (equation (48)) though gauge-fixed is still invariant under the special class of the BRST supertransformations [19], namely:

$$A^a_\mu(x) \to A^a_\mu(x) + (D_\mu\mathcal{C}(x))^a\epsilon \tag{50}$$

$$\mathcal{C}^a(x) \to \mathcal{C}^a(x) - \tfrac{1}{2}f^{abc}\mathcal{C}^b(x)\mathcal{C}^d(x)\epsilon \tag{51}$$

$$\bar{\mathcal{C}}^a(x) \to \bar{\mathcal{C}}^a(x) + \frac{1}{\alpha}(\partial_\mu A^{\mu a}(x))\epsilon \tag{52}$$

where ϵ is an x-independent Grassmann parameter ($\epsilon^2=0$).

It is very important to stress that the BRST invariance of the CS theory ensures satisfying all the Ward–Fradkin–Takahishi–Slavnov–Taylor identities [20], and, therefore, the gauge-invariant renormalizability of the theory [21].

2.1.4. The CSQCD. If one introduces spinor field into the theory to have CSQCD, then it is straightforward to generalize the generating functional equation (48) to this case. The resulting expression is

$$Z[J^a_\mu,\bar{\eta},\eta] = Z_0^{-1}\prod_a\int DA^a_\mu(x)D\bar{\psi}(x)D\psi(x)\exp\Big\{\mathrm{i}(S_{CS}+S_M+S_g+\tilde{S}_\psi) + \mathrm{i}\int \mathrm{d}^3x\,(J^a_\mu(x)A^\mu_a(x)+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x))\Big\}. \tag{53}$$

Here S_{CS}, S_g and S_M were defined earlier, equations (3), (4) and (6), and

$$\tilde{S}_\psi = \int \mathrm{d}^3x\,\bar{\psi}_i(x)(\partial\!\!\!/ + eA\!\!\!/(x) - m)_{ij}\psi_j(x). \tag{54}$$

$i, j = 1,\ldots,N$ above are the colour indices of the $SU(N)$ group in the fundamental representation. It is straightforward to write down the Feynman propagator of the non-Abelian gauge field; it will differ from the Abelian one only by the appearence of colour indices, namely

$$D^{ab}_{\mu\nu}(x) = \delta^{ab}D_{\mu\nu}(x). \tag{55}$$

We would like to emphasize that starting from the generating functionals for the various models that have been considered so far, one can construct all the propagators and the primitive vertices, and thus develop the Feynman rules for perturbation theory. For example, the Feynman propagator for the scalar field is given as

$$G(x-y) = \mathrm{i}\frac{\delta^2 Z[J_\mu,j,j^*]}{\delta j(x)\delta j^*(y)}\mid_{J_\mu=j=j^*=e=0} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3p\,\frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}. \tag{56}$$

Similarly, we have for the spinor propagator from equation (53)

$$S(x-y) = (-\mathrm{i})^2 \frac{\delta^l \delta^r Z}{\bar{\eta}(x)\eta(y)} \,|_{J_\mu=\eta=\bar{\eta}=e=0} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 p \, \frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{\not{p} - m}. \tag{57}$$

2.2. The path integral quantization of pure Chern–Simons quantum chromodynamics by the Batalin–Fradkin–Vilkovisky method

Here, we shall show how to construct the generating functional of the theory of spinors coupled to a non-Abelian gauge field whose action is given solely by the CS term, i.e. pure CSQCD using the BFV method. This will have a more complicated constraint structure than the one with the Maxwell term included. One of the advantages of the BFV quantization method is that it makes the BRST symmetry of the theory more transparent. We start with the classical action:

$$S = S_{CS} + S_\psi \tag{58}$$

where S_{CS} and S_ψ are given by equations (3) and (25). The action can be written in a more transparent form:

$$S_{CS} = -\frac{\mu}{2} \int \mathrm{d}^3 x \, \left(A_0^a(x)\varepsilon_{ij} F^{ija} + \varepsilon_{ij} \dot{A}^{ia}(x) A^{ja}(x) + \frac{g}{3} f^{abc} \varepsilon_{\mu\nu\lambda} A_a^\mu(x) A_b^\nu(x) A_c^\lambda(x) \right) \tag{59}$$

$$S_\psi = \int \mathrm{d}^3 x \, (\bar{\psi}(x)(\mathrm{i}\gamma_0 \partial_0 - \mathrm{i}\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m)\psi(x) - g A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x)). \tag{60}$$

The canonical momenta of the theory turn out to be all primary constraints:

$$\pi_i^a = \frac{\delta \mathcal{L}}{\delta \dot{A}^{ia}} = \frac{-\mu}{2} \varepsilon_{ij} A^{ja} \qquad \theta_i^a \equiv \pi_i^a + \frac{\mu}{2} \varepsilon_{ij} A^{ja} \approx 0 \tag{61}$$

$$\pi_\psi = \frac{\delta^r \mathcal{L}}{\delta \dot{\psi}} = \mathrm{i}\psi^\dagger \qquad \theta_3 \equiv \pi_\psi - \mathrm{i}\psi^\dagger \approx 0 \tag{62}$$

$$\pi_{\psi^\dagger} = \frac{\delta^l \mathcal{L}}{\delta \dot{\psi}^\dagger} = 0 \qquad \theta_4 \equiv \pi_{\psi^\dagger} \approx 0 \tag{63}$$

$$\pi_0^a = \frac{\delta \mathcal{L}}{\delta \dot{A}_0^a} = 0 \qquad G^a \equiv \pi_0^a \approx 0. \tag{64}$$

The standard Poisson brackets are:

$$\{\psi(x), \pi_\psi(y)\} = \{\psi^\dagger(x), \pi_{\psi^\dagger}(y)\} = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{65}$$

$$\{A_\mu^a(x), \pi_\nu^b(y)\} = g_{\mu\nu} \delta^{ab} \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{66}$$

θ_i $(i = 1, 2), \theta_3$ and θ_4 are second class constraints, while G^a is first class. The presence of the second-class constraints motivates one to define the Dirac brackets [22] using these constraints. These can be worked out easily, and the ones that differ from the Poisson bracket are:

$$\{\psi(x), \psi^\dagger(y)\}_D = \mathrm{i}\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{67}$$

$$\{A_i^a(x), A_j^b(y)\}_D = \frac{-1}{\mu} \delta^{ab} \varepsilon_{ij} \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{68}$$

$$\{A_i^a(x), \pi_j^b(y)\}_D = \tfrac{1}{2} g_{ij} \delta^{ab} \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{69}$$

The Hamiltonian assumes the form

$$\mathcal{H} = \mathcal{H}_0 + A_0^a T^a = \bar{\psi}(x)(\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi(x) - \boldsymbol{A}(x)\cdot\boldsymbol{J}(x)$$
$$+A_0^a(x)\left(J_0^a(x) + \frac{\mu}{2}\varepsilon_{ij}F^{ija} + g\frac{\mu}{2}f^{abc}\varepsilon_{ij}A^{ib}(x)A^{jc}(x)\right). \quad (70)$$

$\mathcal{H}_0$ is the Hamiltonian on the constraint surface, and T^a is a first-class constraint analogous to the Gauss' law constraint in QCD, being the generator of the gauge symmetry. A_0 appears here, as is the case in QED and QCD, as a Lagrange multiplier. The first-class constraint T^a can be seen to satisfy the algebra:

$$\{T^a(x), T^b(y)\}_D = -gf^{abc}T^c\delta(x-y) \approx 0 \quad (71)$$
$$\{T^a(x), \mathcal{H}(y)\}_D = 0. \quad (72)$$

The BFV quantization method, in attempting to maintain Lorentz covariance and the unitarity of the S-matrix expands the phase space of the theory by making the Lagrange multiplier of the theory dynamical, and introducing new (ghost) degrees of freedom whose statistics are opposite to the first-class constraints of the theory. In our case we will have two pairs of these ghosts which are Grassmann fields;

$$(\mathcal{C}^a, \bar{\mathcal{P}}^a) \qquad (\mathcal{P}^a, \bar{\mathcal{C}}^a).$$

Therefore, our canonical variables now become

$$Q^A = (A_i^a, \psi, \psi^\dagger, A_0^a, \mathcal{C}^a, \mathcal{P}^a) \quad (73)$$
$$P^A = (\pi_i^a, \pi_\psi, \pi_{\psi^\dagger}, \pi_0^a, \bar{\mathcal{P}}^a, \bar{\mathcal{C}}^a). \quad (74)$$

Generally, the BFV method introduces the so called complete Hamiltonian [16] that enters into the expression of the generating functional, and is defined as

$$\mathcal{H}^{comp} = \mathcal{H}_0 + \{\Psi, \Omega\}_D. \quad (75)$$

Ψ is the gauge fermion of the theory and contains all the gauge degrees of freedom. Ω is the BRST charge of the theory, and satisfies:

$$\{\Omega, \mathcal{H}\}_D = 0 \quad (76)$$
$$\{\Omega, \Omega\}_D = 0. \quad (77)$$

Generally, $\mathcal{H}_l$, Ψ and Ω are found as expansions in powers of the ghost fields by solving equations (76) and (77) above. However, in our case, due to the simplicity of the algebra of the constraints, we get $\mathcal{H}_0$ to zeroth order, Ψ to first order and Ω to second order in the ghost fields. Thus

$$\Psi = \bar{\mathcal{C}}^a\chi^a + \bar{\mathcal{P}}^a A_0^a \quad (78)$$
$$\Omega = \pi_0^b\mathcal{P}^b + T^b\mathcal{C}^b - \tfrac{1}{2}\bar{\mathcal{P}}_b f^{bcd}\mathcal{C}^d\mathcal{C}^c \quad (79)$$

where χ^a is a gauge-fixing function

$$\chi_i^a = \partial_i A_i^a - f^a(x). \quad (80)$$

The vacuum functional of the theory is given now by the expression

$$Z_0 = N\int D\mu(Q,P)\exp\mathrm{i}\left\{\int \mathrm{d}^3x\,(P_A\dot{Q}^A - \mathcal{H}^{\mathrm{comp}})\right\} \quad (81)$$

where P_A and Q_A are given in equations (73) and (74), and

$$D\mu(Q,P) = DA_i^a DA_0^a D\psi D\bar{\psi} D\pi_\psi D\pi_{\psi^\dagger} D\pi_i^a D\pi_0^a D\mathcal{C}^a D\bar{\mathcal{C}}^a D\mathcal{P}^a D\bar{\mathcal{P}}^a$$
$$\times\delta\left(\pi_i^a + \frac{\mu}{2}\varepsilon_{ij}A^{ja}\right)\delta(\pi_\psi - \mathrm{i}\psi^\dagger)\delta(\pi_{\psi^\dagger}(\mathrm{Ber}\|\{\theta_l,\theta_m\}\|)^{\frac{1}{2}}. \quad (82)$$

Ber is the superdeterminant, or the Berezinian, which is introduced here due to the presence of the fermionic degrees of freedom. Integrating over the matter and gauge momenta and over π_0^a, $\bar{\mathcal{P}}^b$ and $\mathcal{P}^a$, we get

$$Z_0 = N \int DA_\mu D\psi D\bar{\psi} D\mathcal{C} D\bar{\mathcal{C}} \delta(\dot{A}_0^a(x) - \partial_i A_i^a(x) + f^a(x)) \times \exp \mathrm{i} \left\{ \int \mathrm{d}^3x \, (\mathcal{L}_{cl} - \bar{\mathcal{C}}^a(\partial_\mu D^{\mu ab} \mathcal{C}^b)) \right\} \tag{83}$$

where

$$\mathcal{L}_{cl} = \mathrm{i}\bar{\psi}(\not{\partial} - m)\psi - A_\mu^a J^{\mu a} - \frac{\mu}{2} A_0^a \varepsilon_{ij} F^{ija} - \frac{\mu}{2} \varepsilon_{ij} \dot{A}^{ia} A^{ja} - g \frac{\mu}{2} f^{abc} A_0^a \varepsilon_{ij} A^{ib} A^{jc} \tag{84}$$

and

$$\bar{\mathcal{C}}^a \partial_\mu D_{ab}^\mu \mathcal{C}^b = \bar{\mathcal{C}}^a (\partial_\mu (\delta_{ab} \partial^\mu - f^{acb} A_c^\mu)) \mathcal{C}^b. \tag{85}$$

The above expression—upon including external sources—coincides with the generating functional equation (53) without the Maxwell term.

3. The S-matrix operator

Although the generating functionals of the theory, equations (10), (29) and (53), contain all the information of the theory, and can be used to derive the scattering amplitudes, it is more convenient to either introduce the path integral representation of the S-matrix of the theory, or to construct the S-matrix operator. The latter is particularly convenient for the investigation of the imaginary parts of the Green functions, Feynman diagrams and the scattering matrix elements, or generally speaking, for the investigation of the unitarity of the theory. We shall first construct the S-matrix operator of the pure CSQED, and then generalize the results to the other cases. A peculiar property of the pure CSQED is the absence of real topological photons, although the propagator and its imaginary part exist (see, for example, equations (20) and (22)). As for the operator $\hat{A}_\mu(x)$; we note that canonical quantization in covariant gauges allows one to introduce (as in QED) operators for the scalar as well as the longitudinal components of $A_\mu(x)$, and it can be proven that due to the canonical commutation relations, the equation for the propagator

$$D_{\mu\nu} = -\mathrm{i}\langle T \hat{A}_\mu(x) \hat{A}_\nu(y) \rangle$$

coincides with the classical equation (13) and they have the same solution, equation (14). However, in the case of pure CS theory, this topological photon does not contribute to the physical states of the Hilbert space, which can be defined as usual; $\partial_\mu \hat{A}_\mu^+ |\mathrm{phys}\rangle = 0$. Thus, starting from this result, one can unambigously formulate rules for the construction of any matrix elements of the different products of this operator. In this sense, one can formulate some kind of Wick theorem for the operators of the topological CS photon. The S-matrix operator for scalar pure CSQED has been constructed in the works [11, 13], and those for spinor pure CSQED in [14]. Here, we would like to elaborate on the constructions given in these references.

In pure CSQED, the S-matrix operator formally has the same form as that in $2 + 1$ dimensional QED,

$$\hat{S} = T \exp\{\mathrm{i} S_{\mathrm{int}}(\hat{\bar{\psi}}, \hat{\psi}, \hat{A})\} \tag{86}$$

here

$$S_{\mathrm{int}}(\hat{\bar{\psi}}, \hat{\psi}, \hat{A}) = \int \mathrm{d}^3x \; : e(\hat{\bar{\psi}} \gamma^\mu \hat{A}_\mu \hat{\psi}) : \tag{87}$$

where ': :' means normal ordering, and $\hat{\psi}(x)$ and $\hat{\bar{\psi}}(x)$ operators are given as

$$\hat{\psi}(x) = \int \frac{d^3 p}{(2\pi)}\sqrt{\frac{m}{E_{\boldsymbol{p}}}}[b(\boldsymbol{p})u(p)e^{-ipx} + d^\dagger(\boldsymbol{p})v(\boldsymbol{p})e^{ipx}] \tag{88}$$

$$\hat{\bar{\psi}}(x) = \int \frac{d^3 p}{(2\pi)}\sqrt{\frac{m}{E_{\boldsymbol{p}}}}[b^\dagger(\boldsymbol{p})\bar{u}(p)e^{ipx} + d(\boldsymbol{p})\bar{v}(p)e^{-ipx}]. \tag{89}$$

$E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and $b(\boldsymbol{p})$ $(d(\boldsymbol{p}))$ and $b^\dagger(\boldsymbol{p})$ $(d^\dagger(\boldsymbol{p}))$ are respectively the annihilation and creation operators of particles (antiparticles) satisfying the usual anticommutation relations:

$$[b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')]_+ = [d(\boldsymbol{p}), d^\dagger(\boldsymbol{p}')]_+ = \delta(\boldsymbol{p} - \boldsymbol{p}'). \tag{90}$$

The two-component spinors $u(p), v(p)$ are, respectively, the positive- and negative-energy solutions of the free Dirac equation in $(2+1)$ dimensions, with the properties:

$$(\hat{p} - m)u(p) = (\hat{p} + m)v(p) = 0 \tag{91}$$

$$\bar{u}(p)u(p) = -\bar{v}(p)v(p) = 1 \tag{92}$$

$$\bar{u}(p)v(p) = \bar{v}(p)u(p) = 0 \tag{93}$$

$$u(p)\bar{u}(p) = \frac{\not{p} + m}{2m} \tag{94}$$

$$v(p)\bar{v}(p) = \frac{\not{p} - m}{2m}. \tag{95}$$

Let us next pay our attention to the operator $\hat{A}_\mu$: Using its above mentioned properties, we can formulate the following rules of the matrix elements of its products: (1) The vacuum expectation value of the products and the T-products of only an even number of the operators A_μ is nonvanishing, and reduces, respectively, to the sum of the vacuum expectation values of the product and the T-product of two field operators defined as

$$\langle 0|T(\hat{A}_\mu(x)\hat{A}_\nu(y))|0\rangle = -iD_{\mu\nu}(x-y) \tag{96}$$

$$\begin{aligned}\langle 0|\hat{A}_\mu(x)\hat{A}_\nu(y)|0\rangle &= -iD^+_{\mu\nu}(x-y)\\ &= -i\int \frac{d^3 p}{(2\pi)^3}\left[\left(\frac{i}{\mu}\varepsilon_{\mu\nu\lambda}p^\lambda - \frac{\alpha}{2}p_\mu\frac{\partial}{\partial^\nu}\right)\delta(p^2)\right]\theta(p_0)e^{ip(x-y)}\end{aligned} \tag{97}$$

where $D_{\mu\nu}(x-y)$ is given by equation (20). For example, for four operator product we have:

$$\begin{aligned}\langle 0|T(\hat{A}_\mu(x)\hat{A}_\nu(y)\hat{A}_\lambda(z)\hat{A}_\delta(u)|0\rangle &= (-i)^2\{D_{\mu\nu}(x-y)D_{\lambda\delta}(z-u)\\ &+D_{\mu\lambda}(x-z)D_{\nu\delta}(y-u) + D_{\mu\delta}(x-u)D_{\nu\lambda}(y-z)\}\end{aligned} \tag{98}$$

and so on. (2) All the matrix elements between physical states of the normal product of any number of the field operators A_μ are equal to zero. However, the vacuum expectation value of the product of the normal products of an equal number of these operators only is different from zero. For example:

$$\begin{aligned}&\langle 0| : \hat{A}_\mu(x)\hat{A}_\nu(y) :: \hat{A}_\lambda(z)\hat{A}_\delta(u) : |0\rangle\\ &\quad = (-i)^2\{D^+_{\mu\lambda}(x-z)D^+_{\nu\sigma}(y-u) + D^+_{\mu\sigma}(x-u)D^+_{\nu\lambda}(y-z)\}\end{aligned} \tag{99}$$

and so on.

Thus, the above rules are the same as the Wick rules except that we take into account the absence of physical states with free topological photons (other than the vacuum state!). Therefore, we now make the following observation. All the Feynman rules of the theory are identical to those of QED given that one replaces the Maxwell propagator in internal lines by the CS propagator, and excludes diagrams with external photon lines. In mathematical language, the above rules mean that the total set of physical states in the total Hilbert space of the theory does not contain states with real free topological photons†, but only the physical states of particles and antiparticles. The interesting consequences and applications of these statements will be considered in section 4.

Consider now the more general case of CSQED, where the propagator is given by equation (14) and the free-field solutions of the classical equations of motion by equation (15). This solution consists of two parts: massive physical part and massless topological part. The canonical quantization of the massive part in the $\alpha = 0$ gauge can be carried out, and gives the following representation for the physical massive part $\hat{A}^m_\mu(x)$ of the operator $\hat{A}_\mu \equiv \hat{A}^m_\mu + \hat{A}^{CS}_\mu$‡

$$\hat{A}^m_\mu(x) = \frac{-1}{2\pi}\int \mathrm{d}^3p\, \mathrm{e}^{\mathrm{i}px}\gamma\left(e^\delta_\mu(p) - \frac{\mathrm{i}}{\mu\gamma}\varepsilon_{\mu\nu\rho}p^\rho e^{\nu\delta}(p)\right)\delta(p^2-\mu^2\gamma^2)a_\delta(p). \tag{100}$$

The S-matrix in this case looks formally the same as (87), but the Wick theorem is now the usual one

$$\langle 0|T\hat{A}_\mu(x)\hat{A}_\nu(y)|0\rangle = -\mathrm{i}D_{\mu\nu}(x-y) + :\hat{A}_\mu(x)\hat{A}_\nu(y): \tag{101}$$

$$\langle 0|\hat{A}_\mu(x)\hat{A}_\nu(y)|0\rangle = -\mathrm{i}D^+_{\mu\nu}(x-y) + :\hat{A}_\mu(x)\hat{A}_\nu(y): \tag{102}$$

$$\begin{aligned}\langle 0|:\hat{A}_\mu(x)\hat{A}_\nu(y)::\hat{A}_\lambda(z)\hat{A}_\sigma(u):|0\rangle &= (-\mathrm{i})^2\{D^+_{\mu\lambda}(x-z)D^+_{\nu\sigma}(y-u)\\ &+D^+_{\mu\sigma}(x-u)D^+_{\nu\lambda}(y-z)\} - \mathrm{i}\{D^+_{\mu\lambda}(x-z)\langle 0|:\hat{A}_\nu(y)\hat{A}_\sigma(u):|0\rangle\\ &+D^+_{\mu\sigma}(x-u)\langle 0|:\hat{A}_\nu(y)\hat{A}_\lambda(z):|0\rangle + D^+_{\nu\lambda}(y-z)\langle 0|:\hat{A}_\mu(x)\hat{A}_\sigma(u):|0\rangle\\ &+D^+_{\nu\sigma}(y-w)\langle 0|:\hat{A}_\mu(x)\hat{A}_\lambda(z):|0\rangle\} + \langle 0|:\hat{A}_\mu(x)\hat{A}_\nu(y)\hat{A}_\lambda(z)\hat{A}_\sigma(u):|0\rangle\end{aligned} \tag{103}$$

and so on, where $D_{\mu\nu}(x-y)$ is given by equation (14). Only one important exception exists. Any matrix element of the normal product of the operators A_μ reduces to that of the normal product of the massive operators A^m_μ;

$$\langle f|:A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n):|i\rangle = \langle f|:A^m_{\mu_1}(x_1)\ldots A^m_{\mu_n}(x_n):|i\rangle. \tag{104}$$

Here, $|i\rangle$ and $|j\rangle$ are two arbitrary physical states of the total Hilbert space of the theory. Now the total set of physical states includes, in addition to spinor particles, real massive photons, but never the topological massless photons.

The generalization of the S-matrix operator to scalar or spinor pure CSQCD is straightforward now. For the spinor case, the generating functional is given by equation (83). The S-matrix will have the form

$$\begin{aligned}S = T\exp\Bigg\{\mathrm{i}\int \mathrm{d}^3x\,\Bigg[&-\mu\varepsilon^{\mu\nu\lambda}\,\mathrm{tr}\left(\frac{2\mathrm{i}}{3}e:\hat{A}_\mu(x)\hat{A}_\nu(x)\hat{A}_\lambda(x):\right)\\ &-\frac{1}{2\alpha}\,\mathrm{tr}(:2e\hat{F}_{\mu\nu}(x)[\hat{A}^\mu(x),\hat{A}^\nu(x)]_- + e[\hat{A}_\mu(x),\hat{A}_\nu(x)]^2_-:)\\ &+e:\partial^\mu\hat{\bar{C}}^a(x)f^{abc}\hat{A}^b_\mu(x)\hat{C}^c(x): + e:\hat{\bar{\psi}}(x)\gamma_\mu\hat{A}^\mu(x)\hat{\psi}(x):\Bigg]\Bigg\}\end{aligned} \tag{105}$$

† The absence of the real topological free photons can be seen most generally from the fact that the CS term does not contribute to the free classical Hamiltonian due to its independence of the metric tensor $g_{\mu\nu}$ in curved spacetime.

‡ The details of the canonical quantization, which is very similar to the Gupta–Bluer quantization will be published in another paper.

The Wick-type theorem for the operators $\hat{\psi}$, $\hat{\bar{\psi}}$, $\hat{\mathcal{C}}$, $\hat{\bar{\mathcal{C}}}$ is as usual. As for the $\hat{A}^a_\mu$ operator, we have the same rules as in the Abelian case, except that the Green function will now have an additional Kronecker delta in the colour indices.

4. Topological unitarity identities

In this part we are going to investigate the consequences of the peculiar property of the CS theories, namely the absence of real topological photons in spite of the presence of the propagator and the many-particle Green function of the gauge field that contribute to the interaction of the particles quantum mechanically (It is well known that on the classical level, the CS fields do not contribute to the interaction of the particles!). We will see that the above property of the CS theories leads, upon imposing the unitarity condition on the theory, to very interesting topological unitarity identities. These identities have been derived in the work [14]. Here, we essentially follow the development in this reference, however, we discuss in more detail how these identities hold in the general case when the Maxwell term is present along with the CS term.

We consider first the case of pure CSQED. The propagator is given by equation (20), and the S-matrix operator is given by equation (86). As we have mentioned above, the absence of the real CS photons means that the complete set of physical vector states in the total Hilbert space of the theory does not contain these topological particles. To investigate the consequences of this fact, we introduce the $\hat{T}$-matrix:

$$\hat{S} = 1 - \mathrm{i}\hat{T} \tag{106}$$

where $\hat{S}$ is the S-matrix operator (the energy–momentum conserving δ-function has been suppressed). The unitarity of the S-matrix operator leads to the well known relation:

$$\mathrm{i}(\hat{T}^\dagger - \hat{T}) = \hat{T}\hat{T}^\dagger = 2\,\mathrm{Im}\,\hat{T}. \tag{107}$$

For arbitrary non-diagonal ($|i\rangle \neq |f\rangle$) on-shell matrix elements between two physical states of the total Hilbert space, we can write the two equivalent relations

$$2\,\mathrm{Im}\langle f|\hat{T}|i\rangle = \langle f|\hat{T}\hat{T}^\dagger|i\rangle \tag{108}$$

and

$$2\,\mathrm{Im}\langle f|\hat{T}|i\rangle = \sum_n \langle f|T|n\rangle\langle n|T^\dagger|i\rangle \tag{109}$$

where in equation (109) we have inserted the complete set of physical states $|n\rangle$ which does not contain the states of the topological photon, but only the states of charged particles. From equation (109) we see that in a given order of perturbation theory, the Feynman diagrams that contribute to the imaginary part on the lhs cannot have intermediate on-shell topological photon lines because $|n\rangle$ are physical states. On the other hand, however, investigating equation (108) in the framework of perturbation theory, we can see that diagrams with intermediate on-shell photon lines do appear since the vacuum expectation value of the product of the normal products of equal number of the operator A_μ (see equation (99)) does not vanish as a consequence of the non-vanishing of the imaginary part of the photon propagator. Therefore, demanding the consistency of equations (108) and (109) leads to the important conclusion that in a given order of perturbation theory, the gauge-invariant sum of the imaginary parts of the Feynman diagrams with a given number of intermediate on-shell photon lines is equal to zero. The vanishing of this sum of the imaginary parts does not mean the vanishing of the sum of the real part, or the vanishing of the imaginary part of each distinct diagram. As a rule, the sum of the real parts of such diagrams will

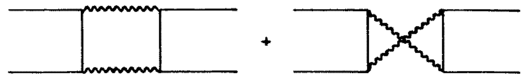

Figure 1. One-loop Feynman diagrams with intermediate topological photon lines contributing to e^+e^- scattering amplitude.

not vanish and will give a contribution to the process involved. Moreover, each diagram in this sum will be an analytic function of invariant variables. The imaginary part of a distinct diagram will vanish only if the diagram is gauge-invariant. These arguments will be demonstrated later when we consider a specific example below.

Now, we illustrate these unitarity identities by an explicit example. Consider the case of scattering of a fermion–antifermion pair in one-loop order in pure CSQED. The S-matrix of this theory is given by equation (86). The gauge-invariant Feynman diagrams with intermediate CS topological photon lines are shown in figure 1. The analytic expression for the imaginary part of each of these diagrams is

$$A_a = \frac{2g^4}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{d}^3k'\left(\delta^+(k^2)\delta^+(k'^2)\delta(p+q-k-k')G_{\mu\lambda}(k)G_{\nu\sigma}(k') \times\frac{\bar{v}(q)\gamma^\nu(\not{p}-\not{k}+m)\gamma^\mu u(p)\bar{u}(p')\gamma^\lambda(\not{p}'-\not{k}+m)\gamma^\sigma v(q')}{((p-k)^2-m^2+\mathrm{i}\epsilon)((p'-k)^2-m^2+\mathrm{i}\epsilon)}\right) \tag{110}$$

$$A_b = \frac{2g^4}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{d}^3k'\left(\delta^+(k^2)\delta^+(k'^2)\delta(p+q-k-k')G_{\mu\lambda}(k)G_{\nu\sigma}(k') \times\frac{\bar{v}(q)\gamma^\nu(\not{k}-\not{q}+m)\gamma^\mu u(p)\bar{u}(p')\gamma^\sigma(\not{p}'-\not{k}+m)\gamma^\lambda v(q')}{((k-q)^2-m^2+\mathrm{i}\epsilon)((p'-k)^2-m^2+\mathrm{i}\epsilon)}\right). \tag{111}$$

where $G_{\mu\nu}(k) = \varepsilon_{\mu\nu\lambda}k^\lambda$, and $\delta^+(k^2) = \theta(k_0)\delta(k^2)$. For simplicity, we restrict ourselves to the case of forward scattering in which case the imaginary parts of these diagrams give their contribution to the total cross section of the process. As was shown in [14], a lengthy calculation gives (an overall irrelevant multiplicative constant has been suppressed)

$$A_a = -\int \mathrm{d}^3k\,\delta^+(k^2)\left(1+\frac{pk}{m^2}+\frac{qk}{pk}\right) = -A_b \tag{112}$$

or

$$A_a + A_b = 0. \tag{113}$$

The same result can be obtained in the case of non-forward scattering too. This example demonstrates the unitarity identities in the one-loop order.

It is not difficult to generalize the unitarity identities to the case when Maxwell-type terms are present. In such cases, one must divide the total gauge-field propagator in equation (14) or (55) into two parts (in the $\alpha = 0$ gauge for example): physical massive part and topological massless part. The operator $\hat{A}_\mu$ in the exponent of the S-matrix in equation (86) can be viewed as the sum of two parts too: the massive physical ($\sim \delta(\gamma^2\mu^2 - k^2)$), and the massless topological part ($\sim \delta(k^2)$). States of the massive photon will appear now in the total Hilbert space of the theory. So, imposing the unitarity condition on this S-matrix in the sense of equations (108) and (109) will lead to the appearence of the topological unitarity identities in this case too.

For example if we consider the diagrams with two intermediate photon lines in the one-loop fermion–antifermion scattering, we get the two unitarity identities illustrated

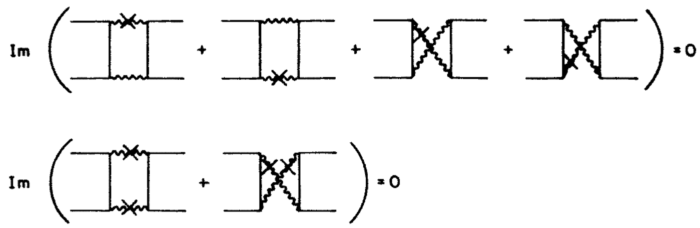

Figure 2. Diagramatic illustration of the unitarity identities in CSQED.

diagramatically in figure 2 (the lines with $\times$ represent the topological part of the gauge-field propagator). The first identity means that the sum of the four diagrams (which is gauge-invariant) with one on-shell intermediate topological photon line is zero. The second identity means the same for the diagrams with two intermediate on-shell topological lines.

The identities developed above can also be shown to hold outside the framework of perturbation theory. That they should hold in the non-Abelian case as well, could be demonstrated without too much difficulty.

5. Concluding remarks

In this paper, we have shown that the covariant path integral quantization of the theories of scalar and spinor fields interacting through the Abelian and non-Abelian pure CS gauge fields, results in a mathematically ill-defined functional integral, since the pure CS action in the exponent of the functional integral is not positive definite in Euclidean space. To define the path integral, it is necessary to introduce into the classical action the Maxwell or Maxwell-type (in the non-Abelian theory) term that is the only bilinear term in the gauge field that does not violate the gauge-invariance of the action. This term also guarantees the gauge-invariant regularization and renormalization of the theory, which then becomes super-renormalizable [2, 3].

The generating functionals of the models considered were constructed, and seen to be formally the same as those of QED (or QCD) in $2+1$ dimensions, with the substitution of the CS gauge-field propagator for the photon (or gluon) propagator. The CS propagator in these models is seen to consist of two parts: the first part is the propagator of a real massive photon (gluon) which contributes to the classical free Hamiltonian, and its states appear in the Hilbert space of the total set of physical states of the system. The second part is that of the topological massless photon which does not contribute to the free Hamiltonian, but leads to additional (in comparison with QED or QCD) interaction between the charged particles. The general solution for the free gauge field, when constructed in a covariant gauge, was therefore seen to consist of a massive part, and a massless topological part.

Taking the limits $\gamma \to \infty$ and $\mu \to 0$ of the propagator and the general solution of the gauge field (see equations (20)–(23)) after renormalization, we get the propagators and the general solutions of the gauge field of the pure CSQED and QED in $2+1$ dimensions, respectively.

Carrying out the path integral quantization of the pure CSQCD using the Batalin–Fradkin–Vilkovisky method, we obtained the same generating functional constructed by the De Witt–Fadeev–Popov method, thus demonstrating the equivalence of the two approaches of quantization of theories with the CS term.

Unfortunately, a path integral representation of the S-matrix is not available for theories with the pure CS field. This is because the 'in' and 'out' limits of the transverse part of the pure CS gauge field do not exist. In the general case, when the Maxwell-type term is included in the action, such a representation can be constructed, and this will depend only on the 'in' and 'out' solutions of the massive part of the gauge field. We constructed in the general case, the *S*-matrix operator for all the Abelian and non-Abelian models, and showed that this operator gives the correct expression for all the Feynman diagrams of the theory, and formally differs from the usual case of QED and QCD in $2+1$ dimensions only by a specific type of Wick theorem for the gauge field.

Starting from this *S*-matrix operator, we have shown that the requirement of the unitarity of the *S*-matrix leads to topolgical unitarity identities that were derived in [14]. These identities demand that at each order of perturbation theory, the gauge-invariant sum of the imaginary parts of the Feynman diagrams with a given number of intermediate on-shell topological photon lines should vanish. These identities were illustrated by some examples in the Abelian case. The importance of these identities stems from the fact that they not only provide additional check of the gauge-invariance of the theory, but also highly facilitate the perturbative gauge-invariant calculations of Feynman diagrams. It is also possible to get strong restrictions on the dependence on invariant variables of the gauge-invariant sum of the real parts of the Feynman diagrams for which the gauge-invariant sum of the imaginary parts vanishes (on account of the analytic properties of Feynman diagrams in the momentum space representation).

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